Topology Vol. 21, No. 4, pp. 393–399, 1982
Printed in Great Britain.

0040–9383/82/040393–07$03.00/0
Pergamon Press Ltd.

COMPACT LIE GROUP ACTIONS ON FINITISTIC SPACES

SATYA DEO and HARI SHANKAR TRIPATHI

(*Received* 23 *October* 1979)

§1. INTRODUCTION

ORBIT SPACES have, once again, become the object of active study after the proof of the Conner Conjecture[9]. Skjelbred[11] has obtained a far reaching generalization of the Conner Conjecture. About the cohomological dimension of the orbit spaces, Quillen's result[10] viz. 'If X is a paracompact Hausdorff space of finite cohomological dimension and G is a compact Lie group acting on X then the cohomological dimension of X/G cannot exceed the cohomological dimension of X', is the best result retaining such an inequality. The spaces X, on which cohomological methods have been used to study the group actions, have been either compact Hausdorff spaces or paracompact Hausdorff spaces of finite cohomological dimension[2, 6]. Swan introduced[12] the concept of finitistic spaces and obtained more general results of the type of classical P.A. Smith fixed point theorems. After that Bredon[3, 4] set the trend of stating results on the cohomological structure of fixed point sets in terms of finitistic spaces. By now it is very clear that finitistic spaces are the right type of spaces to lend themselves for natural generalization of cohomological results about fixed point sets and orbit spaces by the Čech method. However, every time a result is stated for toral actions and rational coefficients, the space X and the orbit space X/T^n are both assumed to be finitistic ([3], pp. 163, 165, 374, 393, 394, 401, 422, etc.). Our main result is the following

THEOREM. *If X is a finitistic space and G a compact Lie group acting on X then the orbit space X/G is finitistic.*

In view of the above theorem the condition that X^* (orbit space) is finitistic can be dropped from all the above theorems [*Ibid*]. This also means that a space is T^k-finitistic iff it is finitistic ([3], p. 164). This can also be regarded as an additional result about orbit spaces. Although it is purely topological, yet it is quite useful in the study of the cohomological structure of fixed point sets and orbit spaces. Another noteworthy point is that in the cohomological study of $Z_p{}^r$-actions with Z_p coefficients, the only topological condition on the space X usually required is that the space X is finitistic ([3], Chap. 3.7). Now in the light of our result, exactly the same condition on X and nothing more will suffice for toral actions with rational coefficients also—a kind of similarity which was known to be valid for compact Hausdorff spaces or finite dimensional (cohomological) paracompact Hausdorff spaces[2, 6].

§2. PRELIMINARIES

Let us recall that an open cover $\{U_\alpha\}$ of a space X is said to be of dimension n (or order $n+1$), $n \geq 0$, if the intersection of any $n+2$ members of $\{U_\alpha\}$ is empty and there exists a subfamily of $\{U_\alpha\}$ of $n+1$ members which has a nonempty intersection. Equivalently an open cover $\{U_\alpha\}$ of space X is of dimension n if each point of X lies in atmost $n+1$ members of $\{U_\alpha\}$ and there exists a point in X lying in some $n+1$ members of $\{U_\alpha\}$. Dimension of any family of subsets of X can be defined in a similar way. We further recall that a space X is said to be of covering dimension $\leq n$, abbreviated to cov dim $X \leq n$, if every finite open covering of X has an open refinement of dimension $\leq n$. We would, at this point, like to remark that we will slightly deviate from the conventional notion of "refinement" just to suit our convenience. If $\{U_\alpha\}$ is a covering of a space X and $\{V_\beta\}$ a family of subsets of X (not necessarily covering of X) such that every member of $\{V_\beta\}$ is contained in some member of $\{U_\alpha\}$ and for some subset A of X, $A \subseteq \cup V_\beta$ then we will still call V_β a refinement of $\{U_\alpha\}$ but, in addition, we will always mention that $A \subseteq \cup V_\beta$ or $\{V_\beta\}$ covers A or $\{V_\beta \cap A\}$ covers A. If $\{V_\beta\}$ is also a covering of X then we will simply say that $\{V_\beta\}$ is refinement of $\{U_\alpha\}$ and nothing more. A space X is said to be of cov dim n if cov dim $X \leq n$ and cov dim $X \not\leq n-1$. If in the definition of covering dimension of a space X we replace "finite open covering" by "open covering" then we have the notion of complete covering dimension of the space X. However, it is well-known that in case of paracompact Hausdorff spaces, the notions of covering dimension and complete covering dimension are equivalent.

Here and elsewhere all our coverings are meant to have been indexed in such a manner that the distinct elements of the indexing set correspond to the distinct members of the covering, unless of course they arise from some construction. Whenever we talk of the dimension of a paracompact Hausdorff space X, abbreviated by dim X, it is always the covering dimension which, if finite, is well-known to be the same as the cohomological dimension [7]. A space is said to be infinite dimensional if it is not finite dimensional.

A paracompact Hausforff space X is said to be finitistic ([3], p. 133) *if each open cover of X has a finite dimensional open refinement.* The concept of finitisticness can be defined in any topological space, but our interest will be only in paracompact Hausdorff spaces. Obviously every compact Hausdorff space is finitistic. It is also obvious that every finite dimensional space is finitistic and that closed subspaces of a finitistic space are finitistic. However, the simplest example of a nonfinitistic space is $\bigcup_{n\geq 0} S^n$, the disjoint union of n-dimensional spheres with weak topology. We give below two examples of nonfinitistic spaces simply because they provide us with a clue for the characterization of nonfinitistic spaces.

Example 1. Let S^∞ be the union of ascending sequence $S^1 \subset S^2 \subset S^3 \subset .. \subset S^n \subset \ldots$ of spheres, $S^n = \{(x_0, x_1, \ldots, x_n) \in R^{n+1} | x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$ with standard embeddings and with the inductive topology. Then S^∞ is a paracompact Hausdorff space, but not finitistic. To see this we first of all observe that for all n the intersection of S^{n+1} with the plane $x_{n+1} = \frac{1}{2}$ is also an n-sphere, say Σ^n which is disjoint from S^n embedded in S^{n+1} in the standard way. Thus the family $\{\Sigma^i_{i\geq 0}\}$ is a family of disjoint spheres such that each Σ^i is closed in S^{i+1}. Since the intersection of $\bigcup_{i\geq 0} \Sigma^i$ with each S^n is closed in S^n, it follows that $\bigcup_{i\geq 0} \Sigma^i$ is closed in S^∞ and therefore each Σ^n is closed in $\bigcup_{i\geq 0} \Sigma^i$. Now for an arbitrary natural number n and for each $k \geq 0$ we define

$$0_{n+k} = \left\{(x_0, x_1, x_2, \ldots, x_{n+1}, x_{n+2}, \ldots, x_{n+k}) \;\middle|\; \begin{array}{l} 1/4 < x_{n+1} < 3/4 \\ -1/8 < x_{n+2} < 1/8 \\ -1/8 < x_{n+3} < 1/8 \\ \quad \cdots \quad \cdots \quad \cdots \\ -1/8 < x_{n+k} < 1/8 \end{array}\right\}.$$

Notice now that $(\bigcup_{j\geq n} 0_j)\cap S^r=\emptyset$ if $r\leq n$ and $(\bigcup_{j\geq n} 0_j)\cap S^r$ is open in S^r, for each $r\geq n+1$. Thus $\bigcup_{j\geq n} 0_j$ is open in S^∞. From the construction of the 0_n's it is also clear that $(\bigcup_{j\geq n} 0_j)\cap(\bigcup_{i\geq 0}\Sigma^i)=\Sigma^n$, which means that Σ^n is also open in $\bigcup_{i\geq 0}\Sigma^i$. We therefore conclude that $\bigcup_{i\geq 0}\Sigma^i$ is the disjoint union of subsets Σ^n such that $\dim \Sigma^n=n$ and each Σ^n is both closed and open in $\bigcup_{i\geq 0}\Sigma^i$. This shows that $\bigcup_{i\geq 0}\Sigma^i$ cannot be finitistic. Since $\bigcup_{i\geq 0}\Sigma^i$ is closed in S^∞, S^∞ cannot be finitistic.

Example 2. Let R^ω denote the countably infinite product of real lines. To see that R^ω, hence any separable Hilbert space of infinite dimension, is not finitistic, we examine the following cubes

$$I^n=\{2n-1\leq x_1\leq 2n, 2n-3\leq x_2\leq 2n-2,\ldots,1\leq x_n\leq 2, x_{n+1}=0=x_{n+2}=\cdots\}.$$

Clearly all of the above cubes are closed subsets of R^ω and are also mutually disjoint. It is also clear that for a given n, we can find an open subset U_n of R^ω such that U_n contains I^n but is disjoint from the rest of the cubes. Thus each I^n is open in the disjoint union $\bigcup_{i\geq 1} I^i$, and it is already observed that each I^n is closed in $\bigcup I^i$. Now to see that $\bigcup I^i$ is closed in R^ω, let $x=(x_1,x_2,\ldots,x_n,\ldots)\notin\bigcup I^i$. Then there exists a coordinate x_j of x such that $x_j\leq 0$ or $x_j\in(2m,2m+1)$ for some integer $m\geq 0$. Consider the product $\prod_{\alpha\in N} X_\alpha=U_j$ where $X_\alpha=R$ for $\alpha\neq j$ and $X_\alpha=(x_j-\epsilon,x_j+\epsilon)$ for $\alpha=j$, ϵ being very small positive number. Then U_j is an open subset of R^ω such that $x\in U_j$ and $U_j\cap(\bigcup I^i)=\emptyset$. Once again we have found a closed subset $\bigcup I^i$ of R^ω such that it is disjoint union of $I^n (n=1,2,\ldots)$, each I^n being closed as well as open in $\bigcup I^i$ and $\dim I^n=n$. Thus $\bigcup I^i$, therefore R^ω, cannot be finitistic.

We have remarked earlier that a closed subspace of a finitistic space is finitistic. However, the countably infinite product $(S^1)^\omega$ of 1-spheres is compact hence finitistic but it has a subspace homeomorphic to R^ω which is not finitistic. The question of product of finitistic spaces is also easily resolved by observing that paracompactness is not preserved under product. Since R^ω is not finitistic it is clear that even if we assume that the product is paracompact Hausdroff, the product of finitistic spaces need not be finitistic; in fact product of even two finitistic spaces need not be finitistic. The next section is devoted to obtain a characterisation of nonfinitistic spaces which will answer the natural question about quotients of finitistic spaces in the form of our main result.

§3. ORBIT SPACES

Since discrete spaces are finitisic, it is clear that an arbitrary continuous image of a finitistic space need not be finitistic. This is also true for any continuous closed image of a finitistic space for the simple reason that the image need not be Hausdorff. Even if it is assumed that the image is Hausdorff, the following example shows that *a continuous closed image of a finitistic space need not be finitistic.* Let $f_2: I\to I^2$, where I is unit interval and I^2 the closed unit square, be a continuous surjective map (Peano Curve). Then by induction define $f_{n+1}: I\to I^{n+1}$ as the compososite $I\xrightarrow{f_2} I\times I\xrightarrow{f_n\times 1} I^n\times I$ for $n\geq 2$. Let $X=\bigcup_{n=1}^{\infty} I_n$ be the disjoint union of countably infinite copies of $I = I_n$ and $Y = \bigcup_{n=2}^{\infty} I^n$ be the disjoint union of cubes. Define $f: X \to Y$ so that $f|I_n = f_n : I \to I^n$. Now f is a continuous closed surjective map, Y is Hausdorff, X is finitistic but Y is not finitistic.

It follows from the General Sum theorem for covering dimensions ([8], p. 193) that if $\{F_n\}$ is a countable closed covering of a paracompact Hausdorff space X such that $\dim F_n \leq k$, for each n, then $\dim X \leq k$. We will make repeated application of this result without mentioning it explicitly. We shall also be using the well-known fact that "If each point x of a paracompact Hausdorff space X has a closed neighbourhood (abbreviated by nbd) N_x with $\dim N_x \leq n$, then $\dim X \leq n$". Now we have the following useful.

LEMMA 3.1. *Let F be a closed subspace of a paracompact Hausdorff space X. Suppose $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ is a family of open sets of X such that $\{U_\alpha \cap F\}$ is an m-dimensional cover of F. Then there exists an open refinement $\{0_\alpha\}_{\alpha \in \mathcal{A}}$ of $\{U_\alpha\}$ such that $\dim\{0_\alpha\} \leq m$ and $\{0_\alpha \cap F\}$ covers F.*

Proof. Consider the open cover $\{U_\alpha\} \cup \{X - F\}$ of X. Let $\{V_\alpha\} \cup \{V_{X-F}\}$ be a precise nbd finite refinement of $\{U_\alpha\} \cup \{X - F\}$ [5, p. 162]. We shrink it further to $\{W_\alpha\} \cup \{W_{X-F}\}$ so that $\bar{W}_\alpha \subsetneq V_\alpha$, for each α and $\bar{W}_{X-F} \subsetneq V_{X-F}$([5], p. 152). Now for each $\alpha \in \mathcal{A}$ we define $\mathcal{A}_\alpha = \{\beta \in \mathcal{A} | \bar{W}_\alpha \cap \bar{W}_\beta \cap F = \emptyset\}$. Let $A_\alpha = \bar{W}_\alpha - \cup\{\bar{W}_\beta | \beta \in \mathcal{A}_\alpha\} = \bar{W}_\alpha \cap (X - E)$, where $E = \cup\{\bar{W}_\beta | \beta \in \mathcal{A}_\alpha\}$ is closed in X, being the union of a locally finite family of closed sets. Now it is clear from the construction that the family $\{A_\alpha\}$ is of dimension $\leq m$, for, $A_\alpha \cap A_\beta = \emptyset$ iff $A_\alpha \cap A_\beta \cap F = \emptyset$. Also $W_\alpha \cap F \subsetneq \bar{W}_\alpha \cap F \subsetneq X - \bar{W}_\beta$ for each $\beta \in \mathcal{A}_\alpha$. Therefore $W_\beta \cap F \subsetneq \bigcap_{\beta \in \mathcal{A}_\alpha} (X - \bar{W}_\beta) = X - \cup \bar{W}_\beta = X - E$. Thus $W_\alpha \cap F \subsetneq W_\alpha \cap (X - E)$ (open) $\subsetneq \bar{W}_\alpha \cap (X - E) = A_\alpha$. Hence $W_\alpha \cap F \subsetneq A_\alpha^0$ (interior of A_α) for each α. This shows that $[0_\alpha\}$, where $0_\alpha = A_\alpha^0$, is as desired.

The above lemma yields the following interesting

PROPOSITION 3.2. *Let X be a paracompact Hausdorff space and F a finitistic closed subspace of X. Suppose each point $x \in X - F$ has a closed nbd N_x in X such that $\dim N_x \leq n$. Then X is finitistic.*

Proof. Let $\{U_\alpha\}$ be an open cover of X. Then, since F is finitistic, we can find a finite dimensional refinement $\{V_\beta \cap F\}$ of $\{U_\alpha \cap F\}$, V_β being open in X, for each β. Again by Lemma 3.1 we can further obtain an open refinement $\{0_\beta\}$ of $\{V_\beta\}$ such that $\{0_\beta\}$ is finite dimensional and $F \subsetneq \cup 0_\beta$. Let $G = X - \cup 0_\beta \subsetneq X - F$. If G is empty there is nothing to prove, because in this case $\{0_\beta\}$ is finite dimensional open refinement of $\{U_\alpha\}$. Let G be nonempty. Then each point $y \in G$ has a closed nbd $M_y = G \cap N_y$ in G such that $\dim M_y \leq n$. Hence $\dim G \leq n$. Now we can obtain, exactly as in case of F, an open refinement $\{W_\gamma\}$ of $\{U_\alpha\}$ such that $G \subsetneq \cup W_\gamma$ and $\{W_\gamma\}$ is finite dimensional. Therefore $\{0_\beta\} \cup \{W_\gamma\}$ is a finite dimensional refinement of $\{U_\alpha\}$.

The next lemma is a step towards the characterisation theorem for nonfinitistic spaces.

LEMMA 3.3 *Let $\mathcal{U}$ be a covering of a space X. Let $F_k(\mathcal{U}) = \{x \in X | x$ lies in atmost k members of $\mathcal{U}\}$. Then (i) If $\mathcal{U}$ is an open covering of X then $F_k(\mathcal{U})$ is closed in X. (ii) If $\mathcal{U}$ is a locally finite closed covering of X then $F_k(\mathcal{U})$ is open in X.*

Proof. Define $\mathcal{U}^n = \{U_1 \cap U_2 \cap \cdots \cap U_n \mid U_i \in \mathcal{U},\ i = 1, 2, \ldots, n\}$ and $\mathcal{U}^{(n)} = \{U_1 \cap U_2 \cap \cdots \cap U_n \mid U_i \in \mathcal{U},\ i = 1, 2, \ldots, n \text{ and } U_i \neq U_j \text{ for } i \neq j\}$. Then $\mathcal{U}^{(n)} \subseteq \mathcal{U}^n$, where $\mathcal{U}^n$ is of course a covering but $\mathcal{U}^{(n)}$ need not be.

(i) Let $\mathcal{U}$ be open covering. Then $\bigcup_{U \in \mathcal{U}} U_{(k+1)}$ is an open set whose complement can easily be seen to be $F_k(\mathcal{U})$.

(ii) Let $\mathcal{U}$ be a locally finite closed covering. Then $\mathcal{U}^{k+1}$ is locally finite closed covering and $\mathcal{U}^{(k+1)}$, being a subfamily of $\mathcal{U}^{k+1}$, is a locally finite family of closed sets. Now $\bigcup_{U \in \mathcal{U}} U_{(k+1)}$ is a closed set whose complement is $F_k(\mathcal{U})$.

Before we are finally equipped to prove our characterisation theorem for nonfinitistic spaces, we need the following.

PROPOSITION 3.4. *Let X be a paracompact Hausdorff space and $\mathcal{U}$ a locally finite open covering of X which is not finite dimensional. Then there exists a sequence $\{F_n\}$ of closed sets of X such that $\cup F_n = X$, $F_n \subsetneq F_{n+1}$ and the family $\{F_{n+1} - F_n\}_{n=1}^{\infty}$ is locally finite. Consequently each point $x \in X$ has a closed neighbourhood N_x in X such that $N_x \subseteq F_m$ for some m. Further, if $\mathcal{U}$ has no finite dimensional open refinement, then in addition, we have* $\dim F_n > n$, *for each n.*

Proof. For every natural number n, we let $F_n = F_n(\mathcal{U})$ as defined in Lemma 3.3. Then clearly $F_n(\mathcal{U}) \subseteq F_{n+1}(\mathcal{U})$, for each n. Also since $\mathcal{U}$ is not finite dimensional, $F_n \neq X$, for each n. $\cup F_n = X$, for, $\mathcal{U}$ is locally finite. Therefore $\{F_n\}$ is nonterminating. Now by re-indexing, if necessary, we can assume that $F_n \subsetneq F_{n+1}$, for each n. Note that as a consequence of re-indexing, $F_n = \{x \in X \mid x$ lies in atmost r_n members of $\mathcal{U}$ for a fixed r_n where $n \leq r_n\}$. Now consider the sequence $\{F_{n+1} - F_n\}_{n=1}^{\infty}$. Clearly $F_{n+1} - F_n = \{x \in X \mid x$ lies in exactly r_{n+1} members of $\mathcal{U}\}$. Therefore it is obvious that if a nbd N_x of x intersects infinitely many members of $\{F_{n+1} - F_n\}_{n=1}^{\infty}$ then it also intersects infinitely many members of $\mathcal{U}$, hence $\{F_{n+1} - F_n\}_{n=1}^{\infty}$ is locally finite. From this it easily follows that every point x in X has a closed nbd $N_x \subseteq F_m$ for some m.

If we further assume that $\mathcal{U}$ has no finite dimensional refinement then in view of the General Sum theorem and the fact that $\dim F_n \leq \dim F_{n+1}$ we can further re-index the sequence $\{F_n\}$ so that $\dim F_n > n$, for each n.

Remark 3.5. One can easily check that if we consider any subsequence $\{F_{n_k}\}_{k=1}^{\infty}$ of $\{F_n\}_{n=1}^{\infty}$ then the family $\{F_{n_{k+1}} - F_{n_k}\}_{k=1}^{\infty}$ is also locally finite.

Now we can prove our characterisation.

THEOREM 3.6. *A paracompact Hausdorff space X is not finitistic iff there exists a closed set G in X which can be expressed as the disjoint union of subsets $\{G_n\}_{n=1}^{\infty}$ such that* $\dim G_n > n$ *and G_n is both closed and open in G for all n.*

Proof. Sufficient part is clear, for, such a G cannot be finitistic and therefore X cannot be finitistic. For the necessary part, suppose X is not finitistic. Let $\mathcal{U} = \{U_\alpha\}$ be a locally finite open cover of X having no finite dimensional open refinement. Then by Prop. 3.4 we have the sequence $\{F_n(\mathcal{U})\}_{n=1}^{\infty}$ of closed sets in X such that $\cup F_n(\mathcal{U}) = X$, $F_n(\mathcal{U}) \subsetneq F_{n+1}(\mathcal{U})$, for each n, $\{F_{n+1}(\mathcal{U}) - F_n(\mathcal{U})\}_{n=1}^{\infty}$ is a locally finite family and $\dim F_n(\mathcal{U}) > n$, for each n. Let $\mathcal{V} = \{V_\alpha\}$ be another open covering of X such that $\bar{V}_\alpha \subseteq U_\alpha$, for each α and let $\bar{\mathcal{V}} = \{\bar{V}_\alpha\}$. Then $F_n(\mathcal{U}) \subseteq F_n(\bar{\mathcal{V}}) \subseteq F_n(\mathcal{V})$ for all n, where by Lemma 3.3, $F_n(\bar{\mathcal{V}})$ is open in X. Consider, for all natural numbers n,

$0_n = \{x \in X \mid x$ has a closed *nbd* N_x in X such that $\dim N_x \leq n\}$. Then one can easily see that 0_n is open in X and $0_n \subsetneq 0_{n+1}$ for all n. Our claim is that $F_r(\mathcal{U}) \cup 0_s \neq X$, for each r and for each s. For this, it is sufficient to prove that $F_{k+1}(\mathcal{U}) \cup 0_k \neq X$, for each k. Suppose, on the contrary, $F_{m+1}(\mathcal{U}) \cup 0_m = X$ for some m. Then $X - F_{m+1}(\bar{\mathcal{V}}) \subsetneq X - F_{m+1}(\mathcal{U}) \subseteq 0_m$ and therefore $\dim(X - F_{m+1}(\bar{\mathcal{V}})) \leq m$. Now by Lemma 3.1 there is an open refinement $\{W_\beta\}$ of $\{V_\alpha\}$ such that $\dim\{W_\beta\} \leq m$ and $X - F_{m+1}(\bar{\mathcal{V}}) \subseteq \cup W_\beta$. From the construction of $F_{m+1}(\bar{\mathcal{V}})$ (see Lemma 3.3 and Proposition 3.4) it follows that $\dim\{V_\alpha \cap F_{m+1}(\bar{\mathcal{V}})\} \leq r_{m+1}$ for some $r_{m+1} \geq m+1$. Therefore we conclude that $\{V_\alpha \cap F_{m+1}(\bar{\mathcal{V}})\} \cup \{W_\beta\}$ is a finite dimensional open refinement of $\mathcal{V}$ hence of $\mathcal{U}$, and this contradicts the fact that $\mathcal{U}$ has no finite dimensional open refinement.

Now let m_1 be the smallest natural number such that $F_{m_1}(\mathcal{U}) - (F_{t_0}(\mathcal{U}) \cup 0_1) \neq \emptyset$, where $t_0 = 1$. This is possible from our claim established above. Let $y_1 \in F_{m_1}(\mathcal{U}) - (F_{t_0}(\mathcal{U}) \cup 0_1)$. Then we can obtain a closed *nbd* G_1 of y_1 and the smallest natural number $t_1 > t_0$ such that $G_1 \cap F_{t_0}(\mathcal{U}) = \emptyset$, $G_1 \subseteq F_{t_1}(\mathcal{U})$ (Proposition 3.4) and $\dim G_1 > 1$. Next suppose m_2 is the smallest natural number such that $F_{m_2}(\mathcal{U}) - (F_{t_1}(\mathcal{U}) \cup 0_2) \neq \emptyset$. For $y_2 \in F_{m_2}(\mathcal{U}) - (F_{t_1}(\mathcal{U}) \cup 0_2)$ we can again find a closed *nbd* G_2 of y_2 and a smallest natural number $t_2 > t_1$ such that $G_2 \cap F_{t_1}(\mathcal{U}) = \emptyset$, $G_2 \subseteq F_{t_2}(\mathcal{U})$ and $\dim G_2 > 2$. Thus by induction we can, for every n, obtain a closed *nbd* G_n of $y_n \in F_{t_n}(\mathcal{U}) - (F_{t_{n-1}}(\mathcal{U}) \cup 0_n)$ and $t_n > t_{n-1}$ such that $G_n \cap F_{t_{n-1}}(\mathcal{U}) = \emptyset$, $G_n \subseteq F_{t_n}(\mathcal{U})$ and $\dim G_n > n$. Thus $\{G_n\}_{n=1}^\infty$, being a precise refinement of the locally finite family $\{F_{t_n}(\mathcal{U}) - F_{t_{n-1}}(\mathcal{U})\}_{n=1}^\infty$ (Remark 3.5), is a locally finite family of mutually disjoint closed sets and therefore $G = \bigcup_{n=1}^\infty G_n$ is the closed set as desired.

Before we are ready to prove our main result we need the following

PROPOSITION 3.7. *Let G be a compact Lie group acting on a finite dimensional paracompact Hausdorff space X. Then* $\dim X/G \leq \dim X$.

Proof. We imitate the proof of Quillen's result[10] which that author proved for the case of cohomological dimension. Let $\dim X = n$. We first recall that if X is a paracompact Hausdorff space and F a closed subspace of X with $\dim F \leq n$ and for every closed subspace A of X such that $A \cap F = \emptyset$, $\dim A \leq n$, then $\dim X \leq n$ ([7], p. 54). Now since the closed subgroups of G satisfy the descending chain condition, we can apply Noetherian induction. Suppose the result is true for any proper closed subgroup G' of G and any G'-space X' (paracompact Hausdorff). Let A be a closed subspace of X/G disjoint from X^G. Then if $q: X \to X/G$ is the orbit map, q can be regarded as orbit map from $q^{-1}(A)$ to A. Now by existence of slice we can, for each $y \in A$, find a closed *nbd* N_y of y in A such that $q^{-1}(N_y)$ admits an equivariant retraction onto $q^{-1}\{y\}$. If $x \in q^{-1}\{y\}$ and G_x is isotropy group of x then $G \times_{G_x} S \xrightarrow{\phi} q^{-1}(N_y)$ defined by $\phi[g, s] = g.s$ is homeomorphism, where S is a closed subset of $q^{-1}(A)$. Thus $q^{-1}(N_y)$ is homeomorphic to $G.S$. But $G.S./G$ is homeomorphic to S/Gx, hence S/G_x is homeomorphic to N_y. Now since G_x is a proper closed subgroup of G, by the induction hypothesis $\dim S/G_x \leq \dim S \leq n$. Therefore $\dim N_y \leq n$. Thus every point of A has a closed *nbd* of $\dim \leq n$, hence $\dim A = \dim q^{-1}(A)/G \leq n$. Again since $\dim X^G \leq n$ and A is arbitrary closed subset of X/G disjoint from X^G (a closed subset of X/G) of dimension $\leq n$, it follows that $\dim X/G \leq n$.

THEOREM 3.8. *Let X be a finitistic space (paracompact Hausdorff) and G a compact Lie group acting on X. Then the orbit space X/G is also finitistic.*

Proof. Suppose X/G is not finitistic. Then by Theorem 3.6 there exists a closed set A of X/G which can be expressed as the disjoint union of subsets $\{A_r\}_{r=1}^{\infty}$ such that $\dim A_r > r$ and A_r is both closed and open in A. Now if $p: X \to X/G$ is the orbit map then by Proposition 3.7 $\dim A_r \leq \dim p^{-1}(A_r)$, for each r. Thus $p^{-1}(A)$ is a closed subset of X which can be expressed as the disjoint union of subsets $\{p^{-1}(A_r)\}_{r=1}^{\infty}$ such that each $p^{-1}(A_r)$ is both closed and open in $p^{-1}(A)$. Hence, again by Theorem 3.6, X cannot be finitistic.

Remark 3.9. For the case of finite group actions, one can verify that our Theorem 3.8 has a simple direct proof. Also in this case one does not have to use the fact that space X is paracompact Hausdorff!

Remark 3.10. As pointed out in the introduction, Quillen's result about the cohomological dimension of the orbit space viz., $\dim_Z(X/G) \leq \dim_Z(X)$ and our analogue of this for covering dimension (Proposition 3.7) are known only for the class of compact Lie groups. Even for compact totally disconnected groups such a result is not true. However, if one can extend such a result for a larger class of groups G even in the form that the difference $|\dim(X) - \dim(X/G)|$ does not exceed a fixed integer, say k, then our main theorem would automatically be valid for such a class of groups also.

REFERENCES

1. G. E. Bredon: The cohomological ring structure of fixed point sets. *Ann. of Maths.* **80** (1964), 524–537.
2. G. E. Bredon: Cohomological aspects of transformation groups. *Proc. Conf. Trans. groups*, pp. 245–280. Springer–Verlag New Orleans (1967). 245–280.
3. G. E. Bredon: *An Introduction to Compact Transformation groups*. Academic Press, New York (1972).
4. G. E. Bredon: Fixed point sets of actions on Poincaré duality spaces. *Topology* **12** (1973), 159–175.
5. J. Dugundji: *Topology*. Allyn and Bacon (1966).
6. Wu-yi Hsiang: *Cohomology Theory of Topological Transformation Groups*. Springer–Verlag, New York (1975).
7. K. Nagami: *Dimension Theory*. Academic Press, New York (1970).
8. J. Nagata: *Modern Dimension Theory*. North-Holland, Amsterdam (1965).
9. Robert Oliver: A proof of the Conner conjecture. *Ann. of Maths.* **103** (1976), 637–644.
10. D. Quillen: The spectrum of an equivariant cohomology ring I. *Ann. of Maths.* **94** (1971), 549–572.
11. T. Skjelbred: Cohomology eigenvalues of equivariant mappings. *Comm. Math. Helv.* **53** (1978), 634–642.
12. R. G. Swan: A new method in fixed point theory. *Comm. Math. Helv.* **34** (1960), 1–16.

University of Jammu
Jammu
180001
India

Ewing Christian College
Allahabad
211003
India